FREE CONVECTION OF HIGHLY VISCOUS LIQUID ALONG A VERTICAL FINITE PLATE WITH CONSTANT HEAT FLUX

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On the basis of asymptotic analysis of the complete Navier–Stokes and energy equations, using the Prandtl number as the basic parameter of the expansion, the form of the velocity profile of free-convective flow is determined at large Prandtl numbers.

Free convection of highly viscous liquids is characterized by the predominant influence of viscous forces over the inertial forces in the flow region. The Prandtl number, which is the ratio of the kinematic viscosity and the thermal diffusivity, determines the thermal boundary layer. Motion outside this layer occurs on account of entrainment of liquid from the surrounding medium.

Insufficiently clear demarcation of these physical laws results in the discrepancy between the theoretical [1] and experimental [2, 3] data for the velocity profiles of the free-convective flow around the vertical surface at large Prandtl numbers. In connection with this, the form of the velocity profiles for a highly viscous liquid is refined in the present work on the basis of asymptotic analysis of the complete Navier–Stokes and energy equations as $\mathrm{Pr} \to \infty$.

Free convection around a vertical plate with a specified constant heat flux at the surface is considered. The coordinate origin lies at the leading edge of the plate. The x axis is directed along the plate and the y axis along the normal to the plate. Dimensionless variables are used: the characteristic dimension L is chosen as the unit of length measurement; the current function is referred to $\nu \mathrm{Gr}^{*3/5}$, and the excess temperature to Lq_w/λ, where ν is the kinematic viscosity, λ is the thermal conductivity, and q_w is the heat flux. Plane steady flow is described by the complete Navier–Stokes and energy equations in the Boussinesq approximation. It is assumed that the work of compression and viscous dissipation of the energy may be neglected [4]. In terms of dimensionless current functions and excess temperature, the corresponding boundary problem takes the form

$$\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\Psi)-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\Psi)=\overset{*}{\mathrm{Gr}}{}^{-2/5}\nabla^4\Psi+\overset{*}{\mathrm{Gr}}{}^{1/5}\frac{\partial\Theta}{\partial y},$$

$$\frac{\partial\Psi}{\partial y}\frac{\partial\Theta}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\Theta}{\partial y}=\mathrm{Pr}^{-1}\overset{*}{\mathrm{Gr}}{}^{-2/5}\nabla^2\Theta, \tag{1}$$

$$\frac{\partial\Psi}{\partial y}=\frac{\partial\Psi}{\partial x}=0,\quad \frac{\partial\Theta}{\partial y}=-1,\ y=0,\ x>0,$$

$$\frac{\partial^2\Psi}{\partial y^2}=\frac{\partial\Psi}{\partial x}=\frac{\partial\Theta}{\partial y}=0,\ y=0,\ x<0, \tag{2}$$

$$\frac{\partial\Psi}{\partial x}=\frac{\partial\Psi}{\partial y}=\Theta\to 0,\ r\to\infty,\ \varphi\neq 0,$$

$$r=(x^2+y^2)^{1/2},\ \varphi=\mathrm{arctg}\,(y/x),\ \mathrm{Gr}^*=\frac{g\beta q_w L^4}{\lambda\nu^2}.$$

The solution of the boundary problem in Eqs. (1) and (2) is undertaken by the method of matchable asymptotic expansions by analogy with [5].

The whole flow region is divided into a thermal boundary layer and an external isothermal region of thickness of the order of $\mathrm{Gr}^{*-1/5}$, in which the viscous-friction force

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predominates over the inertial forces. The solution in the thermal boundary layer is written in the form of an internal asymptotic expansion

$$\Psi(x, y, \overset{*}{\mathrm{Gr}}, \mathrm{Pr}) = \overset{*}{\mathrm{Gr}}{}^{-4/5}\, Pr^{-4/5}\, \Phi_0(x, Y) + \overset{*}{\mathrm{Gr}}{}^{-2/5}\, \overset{-1}{\mathrm{Pr}}\, \Phi_1(x, Y) + \ldots, \tag{3}$$

$$\Theta(x, y, \overset{*}{\mathrm{Gr}}, \mathrm{Pr}) = \overset{*}{\mathrm{Gr}}{}^{-1/5}\, \mathrm{Pr}^{-1/5}\, \Theta_0(x, Y) + \overset{*}{\mathrm{Gr}}{}^{-2/5}\, \mathrm{Pr}^{-2/5}\, \Theta_1(x, Y) + \ldots,$$

which is valid as Pr → ∞ for fixed values of x, $Y = y\mathrm{Pr}^{1/5}\overset{*}{\mathrm{Gr}}{}^{1/5}$. Outside the thermal boundary layer, the solution is written in the form of the expansion

$$\Psi(x, y, \overset{*}{\mathrm{Gr}}, \mathrm{Pr}) \doteq \mathrm{Pr}^{-2/5}\, \Psi_0(x, Z) + \overset{*}{\mathrm{Gr}}{}^{-1/5}\, \mathrm{Pr}^{-3/5}\, \Psi_1(x, Z) + \ldots, \tag{4}$$

which is valid as Pr → ∞ for fixed values of x, $Z = y\overset{*}{\mathrm{Gr}}{}^{1/5}$.

Substituting the expansions of Eqs. (3) and (4) into the boundary problem of Eqs. (1) and (2) and passing to the limit as Pr → ∞ defines a series of boundary problems for the external and internal expansions, interrelated by boundary conditions. Note that the basic parameter in the expansions is the Prandtl number.

In the zero approximation, the external flux remains unperturbed on account of the boundary conditions at infinity. The boundary problem for the thermal layer in the zero approximation using the self-similar transformation

$$\Phi_0(x, Y) = (5x)^{4/5} F_0(\eta),\ \Theta_0(x, Y) = (5x)^{1/5} H_0(\eta),\ \eta = Y(5x)^{-1/5}, \tag{5}$$

takes the form

$$F_3''' + H_0 = 0,\ H_0'' + 4F_0H_0' - H_0 F_0' = 0, \tag{6}$$

$$F_0(0) = F_0'(0) = F_0''(\infty) = H_0(\infty) = 0,\ H_0'(0) = -1.$$

The boundary conditions at the external boundary of the thermal layer are obtained by matching with the external expansion. The first approximation of the external expansion as Pr → ∞ is described by a biharmonic equation with the boundary conditions

$$\nabla^4\Psi_1 = 0,\quad \Psi_1(x, 0) = 0, \tag{7}$$

$$\frac{\partial\Psi_1}{\partial Z}(x, 0) = \begin{cases} (5x)^{3/5}\ F_0'(\infty), & x > 0, \\ 0, & x < 0. \end{cases}$$

This problem has the accurate solution

$$\Psi_1(x, Z) = -Z\frac{F_0'(\infty)}{\sin\dfrac{3\pi}{5}}(5r)^{3/5}\sin\frac{3}{5}(\varphi - \pi). \tag{8}$$

To refine the vertical position of the boundary layer relative to the leading edge, the longitudinal coordinate is deformed in the internal region

$$x = X + f(X, Y)\overset{*}{\mathrm{Gr}}{}^{-1/5}\, \mathrm{Pr}^{-1/5}, \tag{9}$$

and in the external region

$$x = X + f(X, Z)\overset{*}{\mathrm{Gr}}{}^{-1/5}. \tag{10}$$

The deformation function is determined from the condition that the solution in the boundary layer remain self-similar and that the vorticity be zero at x = 0 [5]

$$f(X, Y) = a_1Y + a_0 X^{1/5}. \tag{11}$$

The deformation does not influence the zero approximation of the internal expansion and the first approximation of the external expansion, but complicates the form of the boundary condition for $\Phi_1(x, Y)$

$$\frac{\partial^2\Phi_1}{\partial Y^2}(x, \infty) = \frac{\partial^2\Psi_1}{\partial Z^2}(x, 0) + 2f_Y\frac{\partial^2\Phi_0}{\partial X\,\partial Y}(x, \infty). \tag{12}$$

The system of equations for the first approximation of the internal expansion permits the self-similar transformation

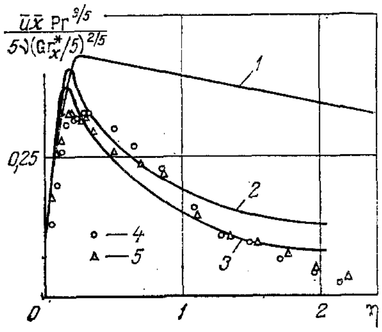

Fig. 1. Velocity distribution in the boundary layer when Pr ≫ 1: 1) boundary-layer theory; calculations from Eqs. (18) and (20) with x = 0.5, Pr = $2 \cdot 10^3$; 2) n = 1; 3) n = 2; 4) experimental results of [2] with Pr = 1700, x = 0.5; 5) with Pr = 2170, x = 0.25.

$$\Phi_1(X,\ Y) = f\,\frac{\partial \Phi_0}{\partial X} + F_1(\eta),\ \Theta_1(X,\ Y) = f\,\frac{\partial \Theta_0}{\partial X} + (5X)^{-3/5} H_1(\eta), \tag{13}$$

where F_1 and H_1 are determined by the system of ordinary differential equations

$$F_1''' + H_1 = 0, \tag{14}$$

$$H_1'' + 4H_1'F_0 + 3H_1F_0' - F_1'H_0 = 0,$$

$$F_1(0) = F_1'(0) = H_1'(0) = H_1(\infty) = 0,\ F_1''(\infty) = -6F_0'(\infty)\,\mathrm{ctg}\,\frac{3\pi}{5}.$$

The process of constructing solutions may be continued up to the second approximation (inclusive). However, the accuracy of the existing experimental data means that it is superfluous to consider the subsequent approximation, and the corresponding results are not given here.

The construction of the internal expansion is not unique. It must be complemented by terms corresponding to the intrinsic solution satisfying the zero boundary conditions

$$c_k \overset{*}{\mathrm{Gr}}{}^{-\frac{1}{5}(\lambda_k+1)}\,\mathrm{Pr}^{-\frac{1}{5}(\lambda_k+4)}\,f_k(\eta)\,(5X)^{-\frac{4}{5}(1-\lambda_k)},$$
$$c_k \overset{*}{\mathrm{Gr}}{}^{-\frac{1}{5}(\lambda_k+1)}\,\mathrm{Pr}^{-\frac{1}{5}(\lambda_k+1)}\,g_k(\eta)\,(5X)^{\frac{4}{5}\left(\frac{1}{5}-\lambda_k\right)}. \tag{15}$$

Then, the following equations are obtained for determining f_k and g_k

$$f_k''' + g_k = 0,$$
$$g_k'' + 4g_k'F_0 + 4(\lambda_k - 1/4)\,g_kF_0' - 4(\lambda_k - 1)\,f_kH_0' - f_k'H_0 = 0. \tag{16}$$

The solution corresponding to the first eigenvalue $\lambda_1 = 5/4$ takes the form

$$f_1(\eta) = 4F_0 - F_0'\eta,\ g_1(\eta) = H_0 - H_0'\eta, \tag{17}$$

where $c_1 = -2a_0$ [5].

From the solution in Eq. (8) for the first approximation of the external expansion in the case of a semiinfinite plate, it follows that infinite velocities exist far from the body. This is associated with an unavoidable singularity at infinity. Therefore, attention now turns to free convection around a plate of finite length, which is also used in experiments. The free-convective flux beyond the trailing edge of the plate is rearranged and in the remote wake takes the form of a plane floating jet. The velocity at the external boundary of the jet is zero [6]. In the transition zone, the damping of the longitudinal velocity may be approximated by a power function, which gives asymptotically correct values.

Consequently, the boundary conditions for a plate of finite length for the first approximation of the external expansion are complicated, but nevertheless an analytic solution may be obtained as a result of integration

$$\Psi_1(x,\ Z)=\frac{ZF_0'(\infty)}{\pi i}\left\{\int_0^1\frac{p^{3/5}dp}{p-t}+\int_1^\infty\frac{p^{-n}dp}{p-t}\right\}, \tag{18}$$

where t = x + iZ.

The expressions for the velocity and temperature in the thermal boundary layer take the form

$$\frac{\lambda(T-T_\infty)}{q_w\bar{x}}=\alpha_0(\overset{*}{\mathrm{Ra}}_x/5)^{-1/5}+\alpha_{10}(\overset{*}{\mathrm{Ra}}_x/5)^{-2/5}+\alpha_{5/4}(\overset{*}{\mathrm{Ra}}_x/5)^{-9/20}+o(\overset{*}{\mathrm{Ra}}_x/5)^{-3/5}, \tag{19}$$

$$\frac{\bar{u}\bar{x}\mathrm{Pr}^{3/5}}{5\nu(\mathrm{Gr}_x^*/5)^{2/5}}=\beta_0+\beta_{10}(\overset{*}{\mathrm{Ra}}_x/5)^{-1/5}+\beta_{5/4}(\overset{*}{\mathrm{Ra}}_x/5)^{-1/4}+O(\overset{*}{\mathrm{Ra}}_x/5)^{-2/5}, \tag{20}$$

$$\alpha_0=H_0(\eta),\ \alpha_{10}=1/5(a_0+a_1\eta)(H_0-H_0'\eta),$$

$$\alpha_{5/4}=-\frac{2a_0}{5^{5/4}}(H_0-H_0'\eta),\ \beta_0=F_0'(\eta),$$

$$\beta_{10}=1/5(a_0+a_1\eta)(3F_0'-F_0''\eta),\ \beta_{5/4}=-\frac{2a_0}{5^{5/4}}(3F_0'-F_0''\eta).$$

The first term in Eqs. (19) and (20) corresponds to the result of boundary-layer theory, the second appears on account of deformation of the longitudinal coordinate, and the third corresponds to the first intrinsic solution.

In Fig. 1, theoretical results are compared with experimental data on the velocity distribution of the free-convective flow when Pr $\gg$ 1. The use of a power law of longitudinal-velocity damping at the external boundary of the thermal layer in the near wake gives a correct interpretation of experimental data on the velocity distribution when n = 2.

The choice of Prandtl number as the basic parameter of the expansion allows an asymptotic theory to be constructed such that the correct form is obtained for the velocity profiles of free-convective flow of a highly viscous liquid, and allows higher-order corrections to the values of the heat-transfer coefficients to be obtained.

NOTATION

x, y, Cartesian coordinate system; Ψ, current function; T, temperature; Pr = ν/a, Prandtl number. Indices: w, wall; ∞, surrounding medium.

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